Investigation of Springs, Hooke's Law, and Spring Oscillations

The force exerted by a spring differs from most forces you have studied in your physics class so far. Most forces you have discussed have been uniform i.e. the forces did not depend on position and did not change as an object moved. In contrast, the force exerted by a spring does depend on position.

Part I: Gaining Intuition for Spring Behavior

We begin by looking at the simplest static situation.

- Open the following simulation:
  - https://phet.colorado.edu/sims/html/hookes-law/latest/hookes-law_en.html
- Choose the "Intro" Option.
- Click the two springs icon on the right side of the screen.
- Check off all five boxes on the top right of the screen.

1. Choose 100 Newtons of applied force for the top spring. (100N is the amount of force equal to the weight of a 10.2kg mass. This is also 22.5 pounds.)
   a. What is the direction of the **applied force**?
   b. What is the direction of the spring force i.e. the force the spring exerts on the pincers? What is the magnitude of the spring force?
   c. Adjust the spring constant until you get a displacement of 0.100m to the right. What is the spring constant?
   d. Is the displacement the same as the length of the spring?
   e. Adjust the applied force until you get a displacement of 0.100m to the left. What is the magnitude and direction of the applied force? What is the magnitude and direction of the force exerted by the spring on the pincers?
   f. Make the force 1/2 the original (50N). By what factor does the displacement change compared to original 0.100? Make the force 1/4 of the original (25N). By what factor does the displacement change compared to the original 0.100m? Make the force 1/5 of the original (20N). By what factor does the displacement change? Make the force 1/10 of the original (10N). By what factor does the displacement change?
   g. Based on your virtual experiment, keeping the same spring constant, and assuming the spring behavior does not change, what would be the displacement with 200N of applied force? What would be the displacement with -0.500N of applied force?
   h. Apply 100N of force to the left to the bottom spring and adjust the spring constant for the bottom spring until you get a displacement of 0.20m to the left (double the displacement of the top spring for the same force). What is the spring constant?
   i. With the same 100N of applied force to the left, find the spring constant to get a displacement of 0.40m to the left. Find the spring constant to get a displacement of 0.50m to the left. Find the spring constant to get a displacement of 1.0m to the left.

j. Based on your virtual experiment, keeping the same 100N of force to the left, and assuming the spring behavior does not change, what would be the displacement for a spring with spring constant 2000N/m? What would be the displacement for a spring with spring constant 50N/m?

2. What is the relationship between the force and the displacement (linear, quadratic, exponential...)?

3. Write down the equation for the applied force, F_{A}, in terms of a spring constant k and the displacement x. Pay attention to the signs.

4. Write down the equation for the force exerted by the spring, F_{SPR}, in terms of a spring constant k and the displacement x. Pay attention to the signs.

Part II: Mass Hanging on a Spring

Now that we are experts in the basic behavior of a spring, let's look at something more interesting.

- Open the following simulation
  - https://phet.colorado.edu/sims/html/masses-and-springs/latest/masses-and-springs_en.html
- Select the "Lab" option. Set damping to "none" on the right side of the screen.
- Check off the first two boxes (for the blue and black lines) in the upper right of the screen. The "natural length" line marks the position of the bottom of the spring when there is no mass hanging off it.
- Pick a mass, place it on one of the springs, and let go. Notice that the motion is **oscillatory**. How can we understand this behavior?

If you hang a mass off the spring, the "mass equilibrium" line is the position of the center of mass when the upwards force that the spring exerts on the hanging mass exactly equals the downward force of gravity exerted on the mass (i.e. the weight of the mass). We know that the force exerted by a spring depends on the position i.e. how stretched it is. That means there is only one position where this equilibrium can occur.

If we hang a mass on the spring and let go at any height other than the equilibrium height, the mass will experience a force, accelerate, and move. Note that when the mass reaches the equilibrium point, it keeps moving. This is because if the spring does not start at the equilibrium position, and only arrives there later, then it arrives at the equilibrium point with a non-zero velocity. This means it keeps moving even though the force on it is momentarily zero.

- Try playing with the spring constant. Does the equilibrium line for a given mass get higher or lower for larger and smaller spring constants?

Let us investigate the case of static equilibrium for a mass hanging on a spring. (Note that the weight is playing the role of the force applied by the pincers from the previous simulation.)

1. Set damping to maximum. Damping is the loss of energy of a spring (or another oscillatory process). Damping in a spring is caused by the internal stress/strain on the material of the spring and air resistance.

2. Place a mass on the spring. Notice that the highly damped spring quickly settles into static equilibrium.

3. On the right side of the screen, above the spring icons, you can pick up a ruler. Use the ruler to measure the displacement from the natural length to the new location of the bottom of the spring. Note that, as mentioned above, the “mass equilibrium” line does not denote the position of the bottom of the spring. It shows the location of the center of mass at equilibrium. For the static equilibrium measurement, the relevant quantity to measure is the displacement of the bottom of the spring. This is the length of green “displacement” arrow.
4. Calculate the weight (i.e. the gravity force) on the mass you chose.
5. Use this information to calculate the spring constant of the spring.
6. Use the slider to change the mass hanging off the spring. Note the new equilibrium line and hang the new mass so it is in equilibrium. Use the new displacement and weight to calculate the spring constant again. Did you get the same thing? Since this is a simulation, the only difference can be from reading the ruler, so your answers should agree to within 5%.
7. Now pick one of the unknown masses and again hang it so it is in equilibrium. Use the ruler and your calculated spring constant to find the unknown mass.

Part III: Mass Hanging on a Spring – Oscillatory Motion

Now that we understand the static equilibrium behavior, let us investigate the oscillatory behavior of springs. **Set damping to “none”.**

First let's make some qualitative observations.

- Think about whether you expect the oscillation to be slower or faster if the spring constant is larger? Try it. Were you right?
    - We expect the oscillation to be faster if the acceleration is larger. We know the spring force increases with the spring constant, so the acceleration increases as well.
- Think about whether you expect the oscillation to be slower or faster on the Moon or on Jupiter, where the gravitational acceleration is much smaller and much larger respectively. Try it. Were you right?
    - We expect the equilibrium point to be lower for Jupiter (large gravity) and higher for the Moon (small gravity). However, because the gravity force is constant, it does not cause the oscillation. To convince yourself of that, you can turn gravity off completely.
    - This also becomes obvious if you consider the fact that you can have a horizontal oscillating spring.
- Think about whether you expect the oscillation to be slower or faster if the mass is larger? Try it. Were you right?
    - We expect the oscillation to be faster if the acceleration is large. We know a larger mass will experience a smaller acceleration with the same spring force so that implies a slower oscillation. We've already discussed that the gravity force is not causing the oscillatory behavior, so its increase does not change the oscillation rate.

Just as we were able to find the spring constant by looking at the equilibrium situation, we can also

find the spring constant using oscillatory motion.

1. Pick a mass and hang it so that an oscillation is produced.
2. Above the spring icons on the right of your screens, you can find a stopwatch which you can move out of the box to make it usable.
3. Measure the period of the spring (i.e. how long it takes the spring to make a complete cycle). You should set the speed of the animation to "slow" to make timing easier.
    a. Pick a specific position (e.g. the equilibrium line) and a specific point on the mass (e.g. the very bottom). Try to start the timer when the two positions line up.
    b. Count 10 (or so) oscillations and try to stop the timer when the two positions are lined up again. You will also get a better result if you time how long several oscillations take (e.g. 10) and divide by the number of oscillations, instead of trying to time a single oscillation.

       Note that a complete cycle looks as follows: the weight starts at some point, moving in some direction, moves to the furthest point, comes back, passes through the start point moving in the opposite direction, moves to the furthest point in this direction, comes back, and passes through the start point again while moving in the same direction as originally.
4. The relationship between the period, the spring constant, and the mass is difficult to figure out without a bit of calculus so we will just look that up.

   $$T^2 = 4\pi^2 \frac{M}{k}$$

   where T is the period, M is the mass, and k is the spring constant.
5. Solve for k and plug in your measurement for T, as well as your value for the mass, M.
6. Do the same thing with another mass. Do the values of k agree with each other? Do the values agree with the value obtained previously? It is reasonable to have some disagreement here (up to 10% percent difference) since you are operating the timer by hand.

Part IV: Approximations

We did not include any damping in our study of oscillatory motion. But we know that in the real world springs do not oscillate forever, because they lose mechanical energy in the form of heat. Although we will not investigate damping in depth,

- Try the lowest nonzero value of damping. How many cycles does it take for the oscillation to die out?

There is a second important way in which this simulation differs from real world spring behavior. The springs are taken to be massless. This results in a significant deviation when the mass if the springs is comparable to the additional mass. This correction changes the above relationship as follows.

$$T^2 = 4\pi^2 \frac{M + \frac{m_s}{3}}{k}$$

where m_s is the mass of the spring. Note the factor of 1/3 multiplying the spring mass.

Part V: Real World Data

Now we will use real world data to determine the spring constant of a spring. Of course, this spring is not massless, but has a mass of 45 grams.

Exercise 1: The first set of data corresponds to the situation in part II, static equilibrium.

Use the data in Table 1 to plot the displacement in meters as a function of the mass, in kilograms. Display the trendline for a linear fit. Since we are plotting the displacement, we know it is 0 for 0 mass, so we can choose the vertical intercept to be 0 (essentially adding a data point for free). Notice that the straight line fits the data points well. (You can display the R^2 value. The closer it is to 1, the better the quality of the fit. The fit is improved by setting the intercept to 0.)

The fact that the spring is massive does not spoil the results. The effect of the mass of the spring in this situation is that when the spring is hung, its length stretches a bit, compared to the length of the spring, when it is laid down. However, since the displacement is measured with respect to this slightly stretched length, the results are unaffected.

$$x = M^*(g/k)$$

So, the slope of your line is g/k. Use this to find the spring constant for the spring. Make sure to state the units.

Table 1

Hanging mass, M, in grams	Hanging mass, M, in kilograms	Displacement, x, in centimeters	Displacement, x, in meters
50		4.0	
100		7.0	
150		11.0	
200		14.0	
250		18.0	

Exercise 2: This data set corresponds to the situation in part III, oscillatory motion.

$$T^2 = \frac{4\pi^2}{k} M$$

Use the data in Table 2 to plot T^2 (in seconds squared) as a function of the mass, in kilograms. Display the trendline for a linear fit. The slope of your line is $4\pi^2/k$. Use this to find the spring constant of the spring. Make sure to state the units.

Compute the percent difference between the spring constant obtained from Table 2 and Table 1. (You should find that it is within a couple of percent.)

We notice that the fit is even better here, but the vertical intercept is not 0. (Setting it to 0, decreases the R^2 value i.e. makes the fit worse.) How can the spring oscillate with 0 mass hanging?

We see that the trendline has the form

$$T^2 = \frac{4\pi^2}{k} M + T_0^2$$

where T_0^2 is the vertical intercept. We now recall the expression that accounts for the spring mass

$$T^2 = 4\pi^2 \frac{M + \frac{m_s}{3}}{k}$$

where m_s is the spring mass. We can see that $T_0^2 = 4\pi^2 \frac{m_s}{3k}$. Use this to calculate the spring mass and show that it is within a few percent difference compared to the given mass, 45 grams.

Table 2

Hanging mass, M, in grams	Hanging mass, M, in kilograms	Average period of oscillation, T, in seconds	Square of period, T^2, in seconds squared
100		0.5641	
150		0.6771	
200		0.7728	
250		0.8583	
300		0.9352	